Method for Shifting Natural Frequencies of Damped Mechanical Systems

Eric K. L. Yee*
Aeronautical Systems Division, Wright-Patterson Air Force Base, Ohio 45433
and
Y. G. Tsuei†
University of Cincinnati, Cincinnati, Ohio 45221

A method for shifting the natural frequency of a damped mechanical system to a desired value is discussed. The solution is derived from the force response equation of the original system. The final equation contains only the degrees of freedom connected to the modifying parameter. Iterations are required, but the solution converges rapidly. Modal coupling effects and sensitivity for shifting a frequency are also derived. Numerical results indicate that the method is effective for system dynamics modification.

Nomenclature

C = damping matrix
$f, f(t)$ = force vector in frequency domain and time domain, respectively
$H(\lambda)$ = complex frequency response function
K = stiffness matrix
M = mass matrix
q = state vector of force f
U = left state modal matrix
V = right state modal matrix
$x, x(t)$ = displacement response vector in frequency domain and time domain, respectively
y = state vector of displacement x
ΔC = change of damping matrix C
ΔC_R = matrix ΔC obtained from changing only one damping R
ΔK = change of stiffness matrix K
ΔK_P = matrix ΔK obtained from changing only one stiffness P
ΔM = change of mass matrix M
ΔM_Q = matrix ΔM obtained from changing only one lumped mass m_Q
δC_R = change of damping R
δk_P = change of stiffness P
δm_Q = change of the lumped mass m_Q at node Q
Λ = diagonal matrix of eigenvalues
λ = complex frequency of excitation; i.e., $\lambda = -\sigma + i\omega$
ω_r = rth damped natural frequency of the system
ω_s = desired damped natural frequency of the system

I. Introduction

METHODS have been introduced in the past for modifying the dynamic characteristics of a mechanical system. Most of these methods predict the modified system dynamics based on the original system characteristics, such as eigenvalues and eigenvectors. Chen and Garba[1] modified the system using an iterative procedure. Berman et al.[2] modified the system by minimizing the error norm between the original and the desired system dynamics. Chou et al.[3] investigated structural dynamics modification using generalized beam mass and stiffness matrices. Elliott and Mitchell[4] discussed the effect of modal truncation and modal modification. Kundra and Nakra[5] discussed the effect of modifying the mass and stiffness matrices to satisfy the changing dynamics of the system. Wang[6] used the reanalysis formulation for structural dynamics modification and optimization. Van Belle[7] and VanHonacker[8] discussed the parameter sensitivity of the structures. Tsuei and Yee[9] discussed a modification method for an undamped mechanical system. This method is hybridized from the modal force technique, a component modal synthesis method investigated by Yee and Tsuei.[10] This paper continues the discussion of the modification method[9] and extends the technique for damped structures. Since the modification of an undamped system was examined thoroughly, the discussion will be focused on damped structures.

II. Force Response of a Damped System

For a damped mechanical system, the equation of motion can be represented as follows:

$$M\ddot{x}(t) + C\dot{x}(t) + Kx = f(t) \tag{1}$$

where M, C, and K can be symmetric or nonsymmetric matrices. The equation can be rearranged and put into the state vector form

$$\begin{bmatrix} O & M \\ M & C \end{bmatrix} \begin{bmatrix} \ddot{x}(t) \\ \dot{x}(t) \end{bmatrix} + \begin{bmatrix} -M & O \\ O & K \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ x(t) \end{bmatrix} = \begin{bmatrix} O \\ f(t) \end{bmatrix} \tag{2a}$$

or in generic form

$$\hat{M}\dot{y}(t) + \hat{K}y(t) = q(t) \tag{2b}$$

where

$$\hat{M} = \begin{bmatrix} O & M \\ M & C \end{bmatrix} \quad \text{and} \quad \hat{K} = \begin{bmatrix} -M & O \\ O & K \end{bmatrix}$$

$$y(t) = \begin{bmatrix} \dot{x}(t) \\ x(t) \end{bmatrix} \quad \text{and} \quad q(t) = \begin{bmatrix} O \\ f(t) \end{bmatrix} \tag{3}$$

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*Systems Engineer, Joint Tactical Autonomous Weapons Systems Program Office.

†Professor, Department of Mechanical, Industrial and Nuclear Engineering. Member AIAA.

The response state vector y can be determined from Eq. (2) and expressed as Eq. (4)

$$y = V[\lambda I - \Lambda]^{-1}U^T q = G(\lambda)q \tag{4}$$

where

$$G(\lambda) = V[\lambda I - \Lambda]^{-1}U^T$$

since

$$y = \begin{bmatrix} \dot{x} \\ x \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ f \end{bmatrix}$$

then

$$\begin{bmatrix} \dot{x} \\ x \end{bmatrix} = \begin{bmatrix} G_{12} \\ G_{22} \end{bmatrix} [f] \tag{5}$$

or simply

$$x = G_{22} f$$

By designating $H = G_{22}$, it can be written as

$$x = Hf \tag{6}$$

It should be noted that if the matrices M, C, and K are symmetric, the left modal matrix U will be the same as the right modal matrix V.

III. Equation of Motion of the Modified System

The equation of motion for a modified system without external excitation can be expressed as

$$(M + \Delta M)\ddot{x} + (C + \Delta C)\dot{x} + (K + \Delta K)x = \mathbf{0} \tag{7}$$

where ΔM, ΔC, and ΔK are the changes of mass, damping, and stiffness of the original system, respectively. Equation (7) can be expanded as

$$M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = -[\lambda^2 \Delta M + \lambda \Delta C + \Delta K]x(t) \tag{8}$$

The right side of Eq. (8) can be treated as external force vector, such that

$$f(t) = -[\lambda^2 \Delta M + \lambda \Delta C + \Delta K]x(t) \tag{9}$$

Combining Eqs. (6) and (9),

$$x = H[-(\lambda^2 \Delta M + \lambda \Delta C + \Delta K)]x \tag{10}$$

IV. Mass, Damping, and Stiffness Matrix for Modification

Consider modifying a lumped mass at mode Q with δm_Q, the matrix ΔM_Q can be expressed as

$$\Delta M_Q = \delta m_Q \begin{bmatrix} 0 & & & & \\ & 0 & & & \\ & & \cdots & & \\ & & & 1 & \\ & & & \cdots & \\ & & & & 0 \end{bmatrix} \tag{11}$$

For a spring with a modification of stiffness δk_p between node I and node J, the matrix ΔK_p can be represented as

$$\Delta K_P = \delta k_p \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & a_{II} & \cdots & 0 & \cdots & a_{IJ} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & a_{JI} & \cdots & 0 & \cdots & a_{JJ} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}_P \tag{12}$$

and

$$A_P = \begin{bmatrix} a_{II} & a_{IJ} \\ a_{JI} & a_{JJ} \end{bmatrix}_P$$

is the stiffness participation matrix for stiffness P. Similarly

$$\Delta C_R = \delta C_R \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & b_{II} & \cdots & 0 & \cdots & b_{IJ} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & b_{JI} & \cdots & 0 & \cdots & b_{JJ} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}_R \tag{13}$$

and

$$B_R = \begin{bmatrix} b_{II} & b_{IJ} \\ b_{JI} & b_{JJ} \end{bmatrix}_R$$

is the damping participation matrix for damping R. Detailed discussion of the ΔM and ΔK were presented by Tsuei and Yee.[9]

V. Shifting Damped Natural Frequency of the Original System to a Desired Value

A. Stiffness Modification for Desired Damped Natural Frequency ω_s

Damped natural frequency can be shifted by changing the mass and stiffness parameters. If only a simple stiffness P is modified, Eq. (10) is reduced to

$$x = H[-\Delta K_P]\, x \tag{14a}$$

or

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = -\delta k_P \begin{bmatrix} h_{1I}(\lambda) & h_{1J}(\lambda) \\ h_{2I}(\lambda) & h_{2J}(\lambda) \\ \vdots & \vdots \\ h_{nI}(\lambda) & h_{nJ}(\lambda) \end{bmatrix} \begin{bmatrix} a_{II} & a_{IJ} \\ a_{JI} & a_{JJ} \end{bmatrix}_P \begin{bmatrix} x_I \\ x_J \end{bmatrix} \tag{14b}$$

Equation (14) is reduced to

$$-\begin{bmatrix} h_{II}(\lambda) & h_{IJ}(\lambda) \\ h_{JI}(\lambda) & h_{JJ}(\lambda) \end{bmatrix} \begin{bmatrix} a_{II} & a_{IJ} \\ a_{JI} & a_{JJ} \end{bmatrix}_P \begin{bmatrix} x_I \\ x_J \end{bmatrix} = \frac{1}{\delta k_P} \begin{bmatrix} x_I \\ x_J \end{bmatrix} \tag{15}$$

It can be considered as

$$J(\lambda)\, z = \gamma\, z \tag{16}$$

where

$$z = \begin{bmatrix} x_I \\ x_J \end{bmatrix}$$

$$J(\lambda) = -\begin{bmatrix} h_{II}(\lambda) & h_{IJ}(\lambda) \\ h_{JI}(\lambda) & h_{JJ}(\lambda) \end{bmatrix} \begin{bmatrix} a_{II} & a_{IJ} \\ a_{JI} & a_{JJ} \end{bmatrix}_P = -H_{IJ}A_P$$

The λ and γ are the unknown parameters in Eq. (16). If the variable λ in Eq. (16) is known, the equation becomes an eigenvalue problem. The solution of this eigenvalue problem is the eigenvalue γ and eigenvector z. Since Eq. (16) is in complex domain, i.e., with real and imaginary parts, its solution γ and z should also be in the complex domain. They are defined as follows:

$$\gamma = \alpha + i\beta \qquad \lambda = -\sigma + i\omega \tag{17}$$

Also note that z vector is of much smaller dimension than x, and it involves only displacement vectors of x_I and x_J that are the subset of vector x.

The $J(\lambda)$ in Eq. (16) is a functional matrix. Each matrix element of $J(\lambda)$ is a polynomial of variable λ and can be expressed as

$$h_{IJ} = \sum_{r=0}^{n} C_{IJ,r}\lambda^r$$

where n is the number of structural modes needed to represent the original system dynamic characteristics within the frequency range of interest.

Determination of δk_P

When a damped natural frequency of the system is shifted from the original value ω_{org} to shifted value ω_s, the imaginary part of λ becomes ω_s, i.e., $\lambda = -\sigma + i\omega_s$. The matrix $J(\lambda)$ cannot be computed because the real part of λ is unknown. As discussed in the last section, Eq. (16) is nonlinear and cannot be solved by direct methods, but λ and γ can be determined using an iterative procedure.

During the iteration, an initial value of σ, denoted as σ_a, is assumed for the real part of λ; the corresponding value of λ is defined as λ_a, i.e., $\lambda_a = -\sigma_a + i\omega_s$. The subscript s indicates that the value of the parameter is the appropriate value for the shifted mode. The subscript a identifies that the value of the parameter is an assumed value and may not be appropriate for the shifted mode; further validation of the value is required. A matrix of $J(\lambda_a)$ can be computed, and Eq. (16) becomes a regular eigenvalue problem:

$$J(\lambda_a)\, z = \gamma\, z \tag{18}$$

The eigenvalue γ and eigenvector z can be determined from Eq. (18). The parametric values of γ and z correspond to λ_a and are denoted as γ_a and z_a.

A modification of stiffness P will result in a δk_P change to the stiffness k_P. Also, the value of δk_P must be a real number for a possible stiffness modification.

From Eqs. (15) and (16)

$$\gamma_s = \frac{1}{\delta k_P} \tag{19}$$

where γ_s are the possible solutions of γ that satisfy Eq. (16). Since δk_P is a real number, γ_s must also be a real number for an acceptable solution of γ. Otherwise, Eq. (19) does not hold. Further, a real value of γ would mean that the parameter β, which is the imaginary part of γ, must be zero.

In essence, two criteria have to be satisfied for a possible solution. The criteria are that Eq. (16) must be satisfied and that the eigenvalue γ must be a real number.

If a value of λ_a is assumed, and the corresponding γ_a value is determined from Eq. (18), but the value of γ_a is not a real number, the values of λ_a and γ_a are not the possible solutions. A different value of λ_a has to be assumed, and the calculation process repeated.

Once the correct values of λ, γ, and z are obtained, the mode shape of the modified system at the desired damped natural frequency ω_s can be calculated from Eq. (4). A summary of the iterative process is in the Appendix.

During the iterative process, the modal damping σ of the shifted mode is continually adjusted until a suitable δk_P is determined. In most cases, the modal damping σ of the shifted mode will be within plus or minus 20% of the σ value of the unshifted mode.

B. Mass Modification for Desired Damped Natural Frequency ω_s

If a mass parameter m_Q is changed, Eq. (10) is reduced to

$$x = -\lambda^2 H \Delta M_Q x$$

or

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = -\lambda^2 \delta m_Q \begin{bmatrix} h_{1Q}(\lambda) \\ h_{2Q}(\lambda) \\ \vdots \\ h_{nQ}(\lambda) \end{bmatrix} [x_Q] \tag{20}$$

Retaining only the x_Q coordinate, it becomes

$$x_Q = -\lambda^2 \delta m_Q\, h_{QQ}(\lambda) x_Q$$

For a particular λ_a, it can be written as

$$\delta m_Q = \frac{-1}{\lambda_a^2 h_{QQ}(\lambda_a)} \tag{21}$$

Similar to the procedures discussed in the stiffness modification section, a desired damped natural frequency ω_s is specified and a value of σ_a is assumed.

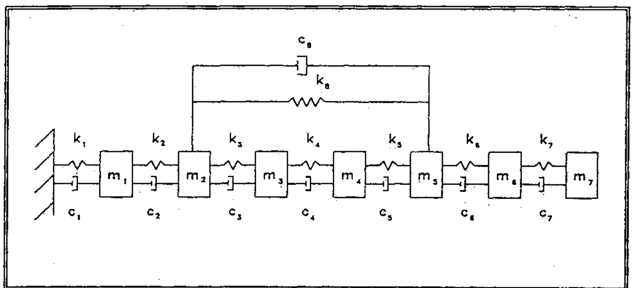
The desired system eigenvalue λ_s is obtained when the imaginary part of $\lambda_a^2 h_{QQ}(\lambda_a)$ is zero [i.e., $\lambda_a^2 h_{QQ}(\lambda_a)$ is real]. The mass change δm_Q at node Q is calculated from Eq. (21). The system mode shape at the frequency ω_s is given in Eq. (20), and it is equal to the Qth column of the frequency transfer function at $\lambda = \lambda_s$.

VI. Identification of Appropriate δk_p

Equation (16) may provide more than one realistic δk_p, but only one δk_p is the correct modification for shifting a particular system mode from original damped natural frequency ω_{org} to a desired value ω_s. The nonunique solutions indicate that the frequency of different modes can be shifted. For instance, there are two possible solutions: one is positive δk_p, and one is negative δk_p. The positive δk_p indicates that the frequency of a lower mode, rth mode, is shifted **up** from ω_r to ω_s, where $\omega_r < \omega_s$. The negative δk_p indicates that the frequency of a higher mode, uth mode, is shifted **down** from ω_u to ω_s, where $\omega_s < \omega_u$. The user has to select the appropriate δk_p to achieve the desired modification. There is a possibility that a system frequency cannot be shifted to a designated value, no matter what δk_p is. In this case, the parameter γ does not converge to a real number during the iterative process. Without a real γ, δk_p does not exist, and this indicates that there is no possible realistic solution.

VII. Coupling of Modes

Another question that has to be addressed is the coupling effects between modes. When the frequency of the rth mode is shifted from ω_r to ω_s with the stiffness modification of δk_p, other modes of the system are also affected. It is always desirable to see where the new uth mode frequency ω_u will be if the rth mode frequency ω_r is shifted to ω_s. To answer this question, modification curves of δk_p vs ω_r and δk_p vs ω_u can be generated and studied. These curves will indicate the modified frequencies of ω_r and ω_u for a particular δk_p. A similar procedure can be applied to other system parameters as well.



r	m_r (Kg)	c_r (N s/m)	k_r (N/m)	Eigenvalue $\lambda_r = -\sigma_s \pm i\omega_r$ Damping $-\sigma_s$ (rad/s)	Natural Frequency ω_r (rad/s)
1	1	0.4	2	0.0111	0.3552
2	1	0.4	2	0.0661	0.9684
3	1	0.2	2	0.2198	1.7400
4	1	0.2	2	0.2366	1.9920
5	1	0.2	2	0.3460	2.3020
6	1	0.3	2	0.4994	2.5250
7	1	0.3	2	0.5211	3.0690
8		0.1	2		

Fig. 1 Configuration of original system.

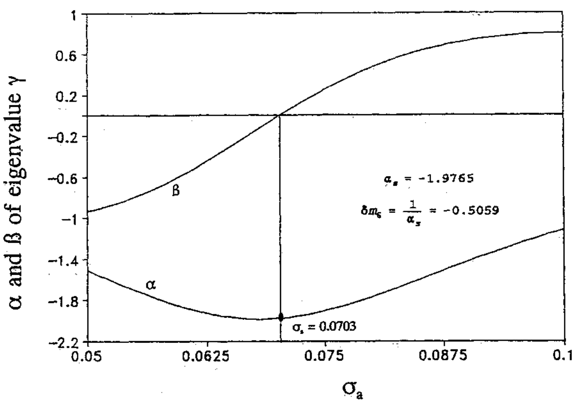


Fig. 2 Determination of α_s required to shift ω_2 from 0.968 to 1.00 rad/s.

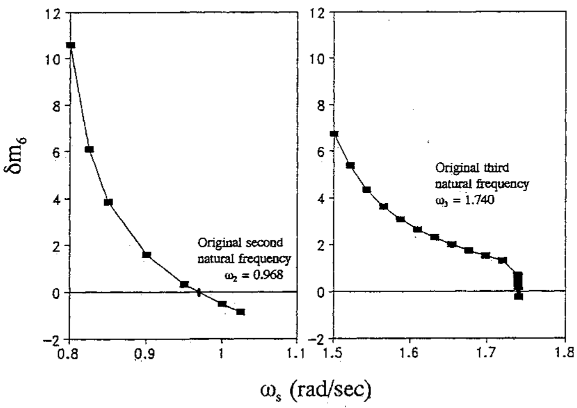


Fig. 3 δm_6 vs the second and third natural frequencies.

The method discussed in the previous sections can be used to generate these modification curves efficiently and to obtain the system modal coupling effects.

VIII. Numerical Examples

A system of seven degrees of freedom is presented for illustration of the modification process. The system configuration is shown in Fig. 1. The second frequency of the original system is 0.968 rad/s. It is desired to shift the second frequency to 1.0 rad/s by modifying the mass m_6. Figure 2 delineates the relationship between σ and γ. When γ becomes real, i.e., $\beta = 0$ and $\gamma_s = \alpha_s + i0$, σ_s is equal to 0.0703. The corresponding α_s is equal to -1.9765. The δm_6 required to shift the frequency to the desired value is $1/\gamma_s$. That is equal to -0.5059. The procedure is carried out for each desired frequency. The δm_6 is plotted against the second and third mode specified frequencies in Fig. 3. From the figure, the δm_6 to

Table 1 Second mode of the modified system $\delta m_6 = -0.506$

Degree of freedom	Modification	Closed form
$\dot{X}_1$	(−0.2448, 0.0000)	(−0.2448, 0.0000)
$\dot{X}_2$	(−0.3676, −0.0073)	(−0.3676, −0.0073)
$\dot{X}_3$	(−0.5456, −0.0212)	(−0.5457, −0.0212)
$\dot{X}_4$	(−0.4496, −0.0134)	(−0.4496, −0.0134)
$\dot{X}_5$	(−0.1277, 0.0101)	(−0.1277, 0.0101)
$\dot{X}_6$	(0.4979, 0.0051)	(0.4980, 0.0051)
$\dot{X}_7$	(1.0000, 0.0000)	(1.0000, 0.0000)
X_1	(0.0171, 0.2436)	(0.0171, 0.2436)
X_2	(0.0185, 0.3663)	(0.0185, 0.3663)
X_3	(0.0171, 0.5444)	(0.0171, 0.5444)
X_4	(0.0181, 0.4483)	(0.0181, 0.4484)
X_5	(0.0190, 0.1263)	(0.0190, 0.1263)
X_6	(−0.0298, −0.4958)	(−0.0298, −0.4958)
X_7	(−0.0700, −0.9951)	(−0.0700, −0.9951)

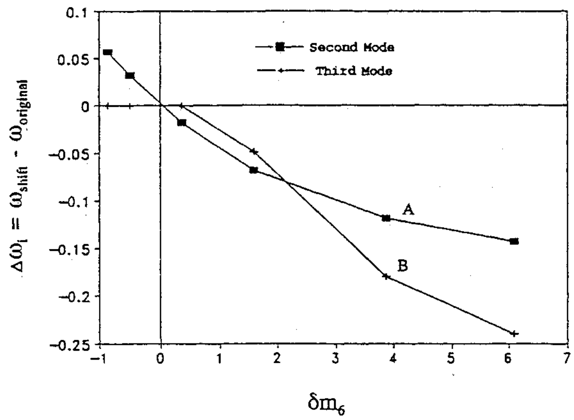


Fig. 4 Coupling effect between second and third modes due to δm_6.

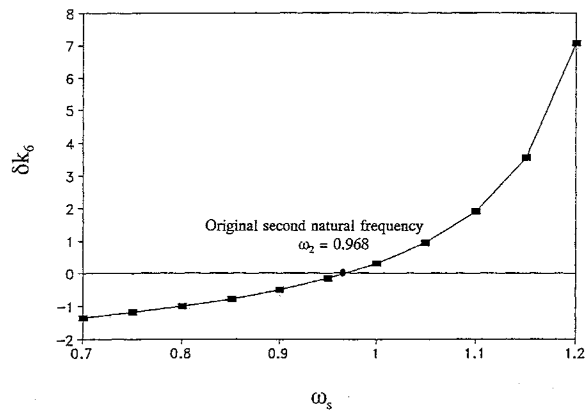


Fig. 5 Modification of stiffness k_6 required for desired second natural frequency ω_s.

be modified for achieving the desired frequency is clearly demonstrated. The coupling effect between the second and third mode are illustrated in Fig. 4. For example, the second frequency is shifted down to 0.118 rad/s labeled as point A. The corresponding shift of the third mode frequency is shifted down 0.18 rad/s as labeled point B.

The mode shapes of the modified system at $\lambda_s = -0.0703 + (1.0)i$ is calculated by the modification process and also by closed-form solution. Both mode shapes are tabulated in Table 1. As can be seen, the mode shapes are identical. A similar approach is also used to shift the second system frequency by modifying the stiffness k_6. The δk_6 vs ω is plotted in Fig. 5. It should be noted that, unlike other methods,[7,8] the derivatives of damped natural frequency ω_s with respect to k or m are not required.

IX. Conclusion

A method for modifying a mechanical system and shifting its damped natural frequency to a desired value has been presented. The calculation is based on the original system dynamic characteristics. The degrees of freedom in the final equation are much smaller than the degrees of freedom of the entire structure. Even though the calculation involves iteration, it converges rapidly and normally requires only a few iterations. The δk and δm vs desired damped natural frequency can be plotted. These curves can be used to understand the effectiveness of δk_p and δm_Q on changing the damped natural frequency of the system. The results indicate that the method is effective and computationally efficient for system dynamics modification.

Appendix: Summary of the Iterative Process

The iterative process can be summarized as follows:

1) Specify the desired damped natural frequency ω_s.

2) Determine the initial value of σ, denoted as σ_a, and the step increment of σ, denoted as $\delta\sigma_a$.

3) Define λ_a, where $\lambda_a = -\sigma_a + i\,\omega_s$.

4) Calculate matrix $J(\lambda_a)$ from Eq. (16).

5) Determine eigenvalue of γ from Eq. (16).

6) If $|\beta| <$ tolerance, go to step 9; otherwise, go to step 7. β is the imaginary part of γ.

7) If the value of $\delta\sigma_a$ was changed during last iterative cycle, increase σ_a to $\sigma_a + \delta\sigma_a$, and go to step 3. Otherwise, do not increase the value of σ_a, and go to step 8.

8) Compare the previous iterative cycle β value, denoted as $\dot{\beta}$, with current iterative cycle β value, denoted as $\bar{\beta}$:

Condition 1

$$\dot{\beta} \times \bar{\beta} > 0$$

Increase the value of σ_a to $\sigma_a + \delta\sigma_a$, go to step 3.

Condition 2

$$\dot{\beta} \times \bar{\beta} < 0$$

Decrease the value of σ_a to $\sigma_a - \delta\sigma_a$, decrease the value of $\delta\sigma_a$ to $(\delta\sigma_a)/2$, and go to step 3.

9) Calculate eigenvector z from Eq. (16).

10) Calculate δk_P from Eq. (19).

11) Calculate mode shape from Eq. (14).

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